Assignment 2.

This homework is due *Thursday*, September 13.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Quick reminder

A relation $R \subseteq X \times X$ is an *equivalence* relation if it is

- reflexive: $\forall x \in X \ xRx$,
- symmetric: $\forall x, x' \in X$ if xRx' then $x'Rx$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A relation $R \subseteq X \times X$ is a *partial ordering* (partial order) if it is

- reflexive: $\forall x \in X \ xRx$,
- antisymmetric: $\forall x, x' \in X$ if xRx' and $x'Rx$ then $x = x'$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A partial order is called *total* if $\forall x, x' \in X \ xRx'$ or $x'Rx$.

2. Exercises

(1) Determine whether the following are equivalence relations on X:
   (a) $X = \mathbb{R}$, $x \approx y$ if and only if $|x - y| < 0.1$.
   (b) $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, $(n_1, m_1) \equiv (n_2, m_2)$ if and only if $n_1 m_2 = m_1 n_2$. (What is a good name for this? There are at least two answers.)
   (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$ (the set of functions from a subset of $\mathbb{R}$ to $\mathbb{R}$),
   functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ "partially agree" if and only if their restrictions on $A \cap B$ are equal:
   $$f \sim g \quad \text{iff} \quad f|_{A\cap B} = g|_{A\cap B}.$$

(2) Describe all relations that are equivalences and total orderings at the same time.

(3) Determine whether the following are partial orders on X:
   (a) $X = \mathbb{R}_{>0}$ (positive reals), $x \ll y$ if and only if $y/x > 10$.
   (b) $X = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ (closed intervals), $[a, b] \preceq [c, d]$ if and only if $a \leq c$ and $b \leq d$.
   (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$. For functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, put $f \preceq g$ if and only if
   $A \subseteq B$ and $f = g|_A$; in other words, if and only if f is a restriction of g.

(4) (1.4.27) Is the set of rational numbers open? Is it closed?

— see next page —

(5) (1.4.28) Prove that $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed. (*Hint:* There are many ways to approach this. One of them is to use classification of open sets in $\mathbb{R}$.)
Comment. Those familiar with general topology may see that this problem asks to prove that $\mathbb{R}$ is connected.

(6) (1.4.36) Show that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra $\mathcal{A}$ that contains all intervals of the form $[a, b)$, where $a < b$. (*Hint:* Show that *both* $\mathcal{B}$ and $\mathcal{A}$ contain *both* open intervals and intervals of the form $[a, b)$.)

(7) (1.4.37) Show that every open set in $\mathbb{R}$ can be represented as a countable union of closed sets.

3. Extra exercise

Problem below will only go to the numerator of your grade for this homework. Also, the due date on this problem is December, 7. That is, you can submit this problem any time before classes end.

(8) Suppose X and Y are two sets. Prove that if there is an injection $f : X \to Y$, and an injection $g : Y \to X$, then there is a bijection $\varphi : X \to Y$.
Comment. This problem essentially asks to prove that "injects into" is a partial order on classes of equipotence. Don't remember if I said this word in class, but two sets are called *equipotent* if there is a bijection between them.